Closed-Form Likelihood Functions for SCR

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[*]University of Auckland

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[1] https://www.roamingowls.com

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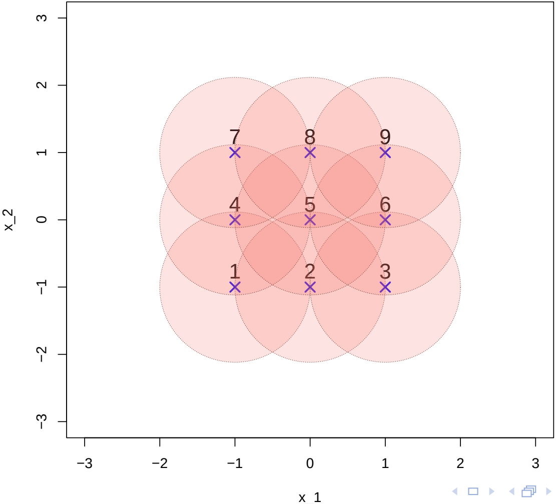

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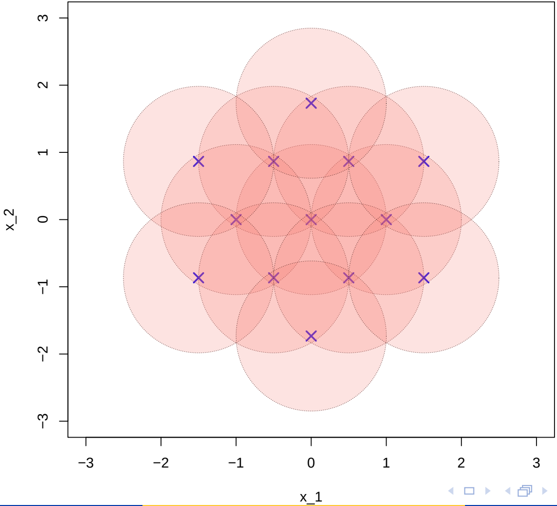

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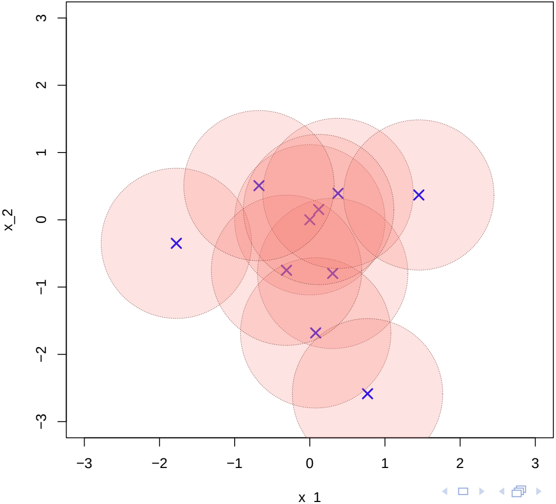

Assumption I

[4] Image credits: Ilia Shalamaev

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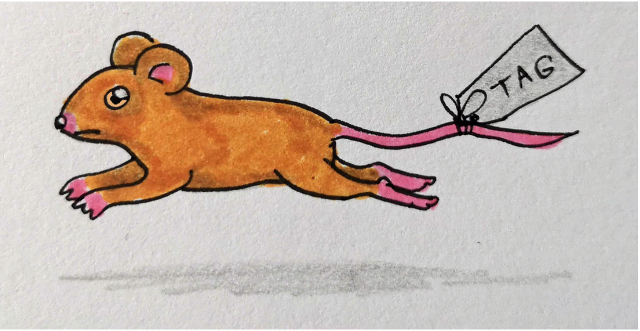

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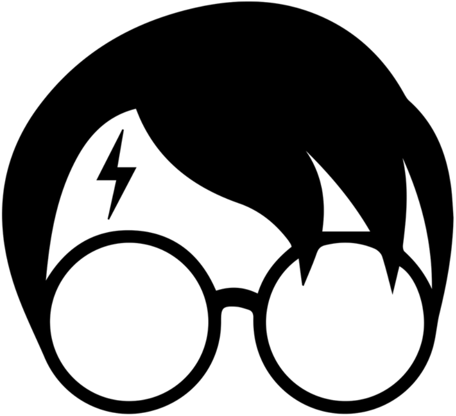
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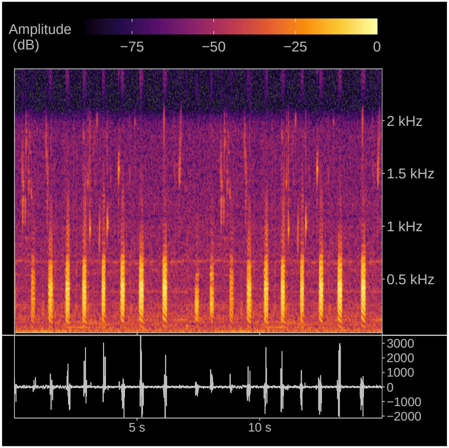

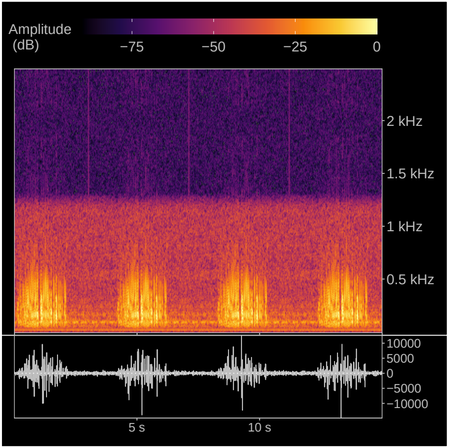

Data

- In its simplest form, our survey data typically look like the following.

Animal ID	Animal Name	Binary Capture History at each Detector						
		d1 (-1,-1)	d2 (0,-1)	d3 (1,-1)	d4 (-1,0)	d5 (0,0)	d6 (1,0)	d7 (-1,1)
1	Homer	1	1	0	1	1	1	0
2	Marge	1	1	0	1	0	0	1
3	Lisa	1	1	1	0	0	1	1
4	Bart	0	0	0	1	1	1	0
⋮								
$N-3$	Burns	0	0	0	0	1	0	0
$N-2$	Apu	0	1	0	1	0	1	1
$N-1$	Krusty	1	0	0	1	1	0	1
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Table: Capture History

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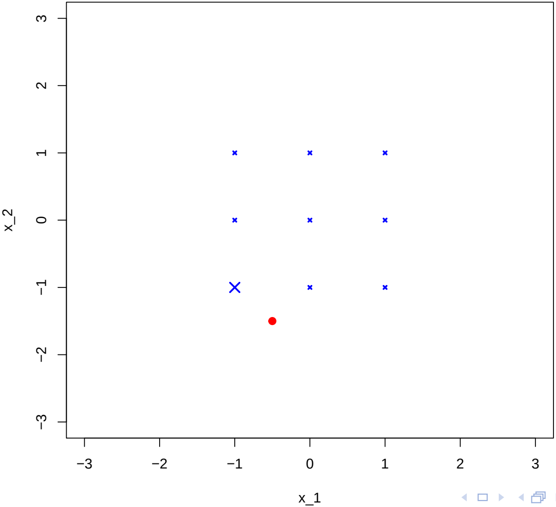

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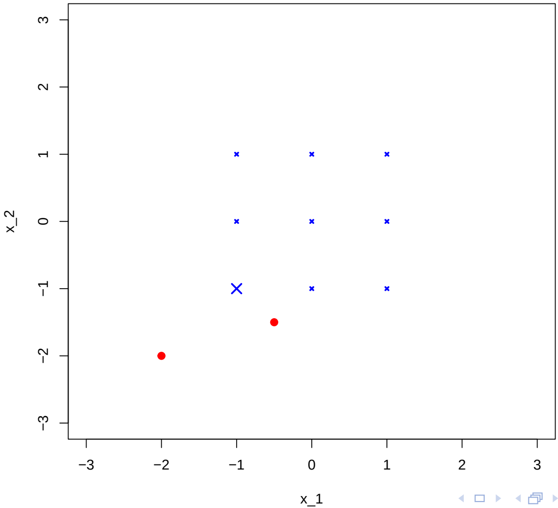

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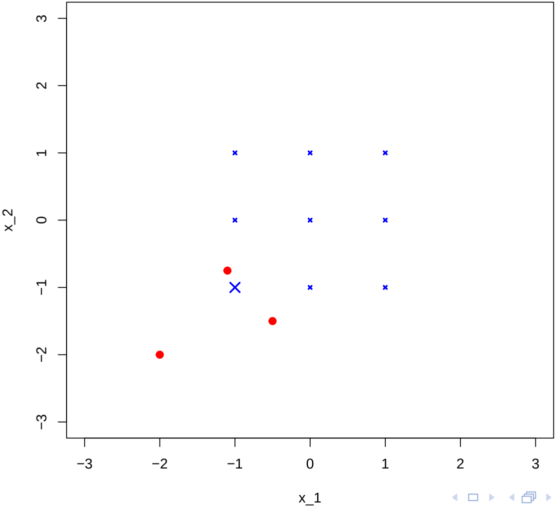

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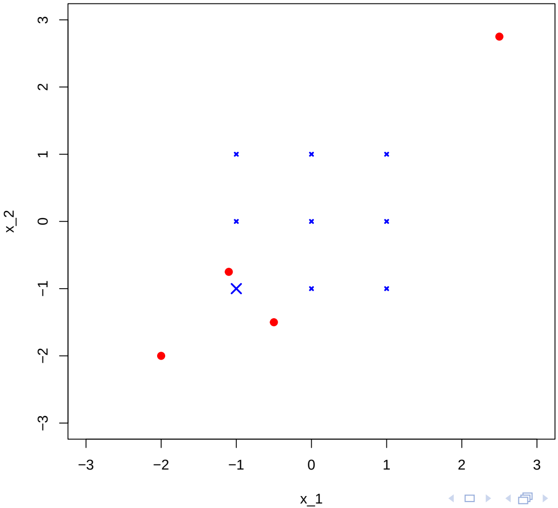

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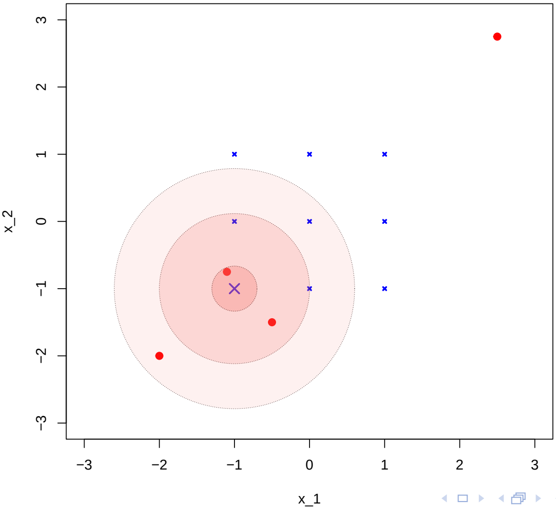
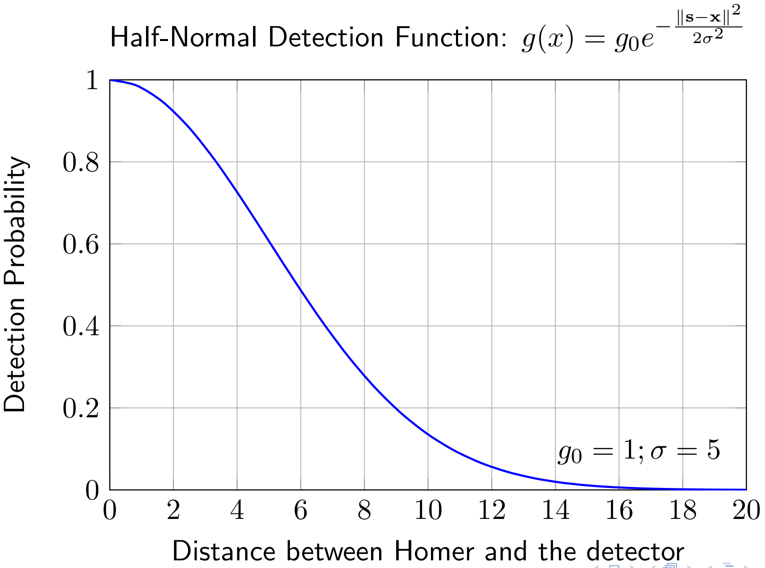


Detection Function



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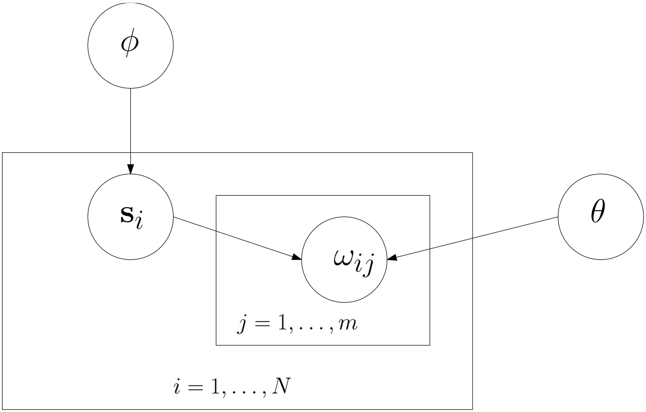

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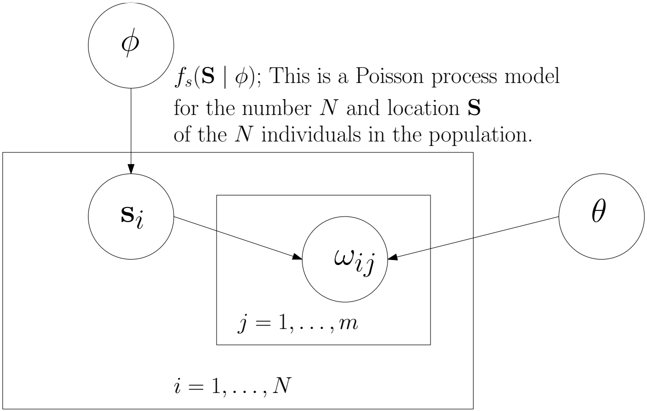

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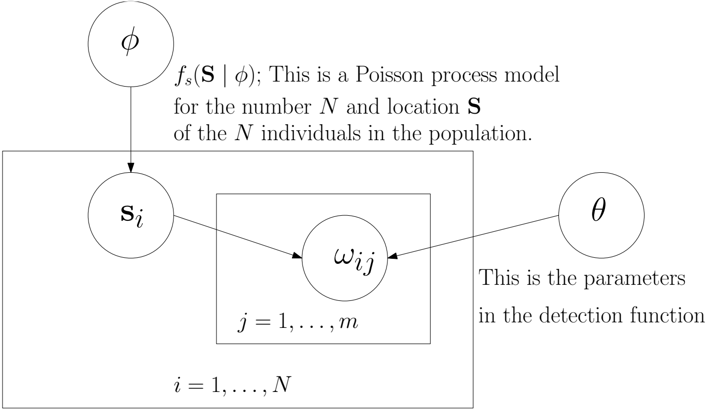

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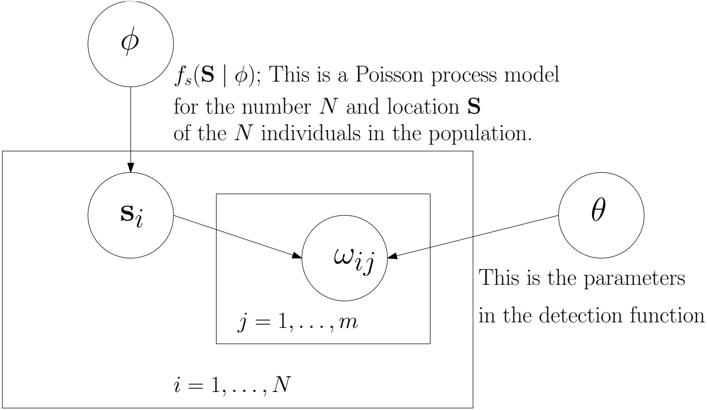

$f_\Omega(\Omega \mid \mathbf{S}, \mathbf{X}, \theta)$; This is the probability model for the capture histories Ω, given individuals' locations, $\mathbf{S}$ and detectors' locations, $\mathbf{X}$.

Unusual Scenario-Complete data

- If *all* N *individuals were detected*, given the locations $\mathbf{S}$ and $\mathbf{X}$, then

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  where $\boldsymbol{\theta} = (g_0, \sigma^2)$,

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where $\boldsymbol{\theta} = (g_0, \sigma^2)$, and $g_j = g_0 \exp\left(-\frac{\|\mathbf{s}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$.

- Assume independence between individuals and detectors.
- Recall the capture history is binary, and the detection is halfnormal.

Assumption II

- If we force the Poisson point process to be homogeneous, then $\theta = D$,

$$f_s\left(\mathbf{S} \mid D\right) = \frac{\left(D \cdot A\right)^N \exp\left(-D \cdot A\right)}{N!} \prod_{i=1}^{N} \frac{1}{A}$$

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- Conditioning on location $\mathbf{S}$, we have the following likelihood

$$\begin{aligned} \mathcal{L}^c &= f_s(\mathbf{S} \mid D) \cdot f_\Omega(\Omega \mid \mathbf{S}, \boldsymbol{\theta}) \\ &= \frac{D^N \exp(-D \cdot A)}{N!} \prod_{i=1}^{N} \prod_{j=1}^{m} g_j^{\omega_{ij}} \cdot (1 - g_j)^{1-\omega_{ij}} \end{aligned}$$

  which is known as complete-data likelihood (King et al, 2016).

Latent variable

- Of course, the location $\mathbf{s}_i$ in practice is a latent variable,

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- We have to marginalise over s before we can estimate D, g_0 and σ.

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \mathcal{L}^c \left(\boldsymbol{\theta}; \boldsymbol{\Omega}, \mathbf{S}, \mathbf{X}, N\right) \, d^2\mathbf{s}_1 d^2\mathbf{s}_2 \cdots d^2\mathbf{s}_N$$
$$= \frac{D^N \exp\left(-D \cdot A\right)}{N!} \prod_{i=1}^{N} \int_{\mathbb{R}^2} \prod_{j=1}^{m} g_j^{\omega_{ij}} \cdot \left(1 - g_j\right)^{1-\omega_{ij}} \, d^2\mathbf{s}$$

Unobserved animals

- Of course, some animal would evade detection, then the following

$$\prod_{j=1}^{m}(1-g_j)=\prod_{j=1}^{m}\left\{1-g_0\exp\left(-\frac{\|\mathbf{s}-\mathbf{x}_j\|^2}{2\sigma^2}\right)\right\} \tag{1}$$

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- The probability of the animal at $\mathbf{s}$ detected by at least one detector is

$$p(\mathbf{s})=1-\prod_{j=1}^{m}\left(1-g_j\left(\mathbf{s}\right)\right) \quad (2)$$

- Integrating p over $\mathbf{s}$ gives the total probability of detecting an animal.

$$F_{\mathsf{esa}}\left(g_0,\sigma^2,\mathbf{X}\right)=1-\int_{\mathbb{R}^2}\prod_{j=1}^{m}\left(1-g_j\left(\mathbf{s}\right)\right)d^2\mathbf{s} \quad (3)$$

Truncation and thinning

- In this case, we have the conditional probability of observing capture history Ω conditioning on detecting n number of individuals.

$$f_{\Omega}\left(\Omega \mid \mathbf{S}, \boldsymbol{\theta}\right) = \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{g_j\left(\mathbf{s}\right)^{\omega_{ij}} \cdot \left(1 - g_j\left(\mathbf{s}\right)\right)^{1-\omega_{ij}}}{F_{\text{esa}}} \tag{4}$$

- And instead of working with a Poisson point process for the number of animals, we have to work with the point process for the number of detected animals, which is also Poisson,

$$\mathcal{L} = \frac{D^n \exp\left(-D \cdot F_{\text{esa}}\right)}{n!} \prod_{i=1}^{n} \int_{\mathbb{R}^2} \prod_{j=1}^{m} g_j^{\omega_{ij}} \cdot \left(1 - g_j\right)^{1-\omega_{ij}} \, d^2\mathbf{s} \tag{5}$$

Notation

- We need to solve two integrals before obtaining the closed-form likelihood,

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- Let $[m] = \{1, 2, \cdots, m\}$ denote the set of the first m natural numbers, and

$$\mathcal{S}_k = \{\mathcal{B} \in \mathcal{P}\left([m]\right) \mid |\mathcal{B}| = k\}$$

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- That is, $\mathcal{S}_k$ is the set of all k-combinations of $[m]$, e.g., if $m = 3$, then

$$\mathcal{S}_1 = \big\{\{1\}, \{2\}, \{3\}\big\}$$
$$\mathcal{S}_2 = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}\big\}$$
$$\mathcal{S}_3 = \big\{\{1, 2, 3\}\big\}$$

- The halfnormal detection function is separable,

$$g_j = g_0 \exp\left[-\frac{\|\mathbf{s} - \mathbf{x}_j\|^2}{2\sigma^2}\right], \quad \text{for } j = 1, \ldots, n_d, \tag{7}$$

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$$= -\sum_{k=1}^{m} \sum_{\mathcal{B}\in\mathcal{S}_k} (-1)^k \underbrace{\int_{\mathcal{R}} \prod_{j\in\mathcal{B}} a_j \cdot b_j\ d^2\mathbf{s}}_{\text{Separable}} \tag{11}$$

How can we interpret this result?

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$$\mathrm{P}\left(\bigcup_{j=1}^{3} E_i\right) = \sum_{i=1}^{3} \mathrm{P}\left(E_i\right) - \sum_{i,j\in\mathcal{S}_3; i\neq j} \mathrm{P}\left(E_i \cap E_j\right) + \mathrm{P}\left(E_1 \cap E_2 \cap E_3\right)$$

- Reducing double to single by separating the integrals, we have

$$\int\limits_{\mathcal{R}} \prod_{j \in \mathcal{B}} g_j \, d^2\mathbf{s} = \underbrace{\int\limits_{\mathcal{R}} \prod_{j \in \mathcal{B}} a_j \, ds_1}_{\alpha_{\mathcal{B}}} \cdot \underbrace{\int\limits_{\mathcal{R}} \prod_{j \in \mathcal{B}} b_j \, ds_2}_{\beta_{\mathcal{B}}} \tag{12}$$

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where $\alpha_{\mathcal{B}}$ and $\beta_{\mathcal{B}}$ can be found, by using integration by parts, regrouping, and results in Gaussian integrals, which lead us to the following form

$$\gamma_{\mathcal{B}} = \int_{\mathbb{R}^2} \prod_{j \in \mathcal{B}} g_j \, d^2\mathbf{s} = g_0^{|\mathcal{B}|} \exp\left[\frac{|\mathcal{B}|}{2\sigma^2} \left(\|\mathbf{c}_{\mathcal{B}}\|^2 - \mu_{\mathcal{B}} \right) \right] \frac{2\pi\sigma^2}{|\mathcal{B}|} \tag{13}$$

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- The term $\mathbf{c}_{\mathcal{B}}$ denotes the centroid of detectors defined by the set $\mathcal{B}$ and $\mu_{\mathcal{B}}$ denotes the mean squared Euclidean norm of the detector coordinates

$$\mu_{\mathcal{B}} = \frac{1}{|\mathcal{B}|} \sum_{j \in \mathcal{B}} \|\mathbf{x}_j\|^2 \tag{14}$$

Detection integral

- The other type of integrals can be found in a similar way

$$\mathcal{L} = \frac{D^n \exp\left(-D \cdot F_{\text{esa}}\right)}{n!} \prod_{i=1}^{n} \int_{\mathbb{R}^2} \prod_{j=1}^{m} g_j^{\omega_{ij}} \cdot \left(1 - g_j\right)^{1-\omega_{ij}} \, d^2\mathbf{s} \tag{15}$$

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which means we never need to deal with any integral in high dimension.

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- Using the above notation and a similar strategy, we can rewrite

$$F_{\text{det}} = \prod_{i=1}^{n} \int_{\mathbb{R}^2} \prod_{j=1}^{m} g_j^{\omega_{ij}} (1 - g_j)^{1-\omega_{ij}} \, d^2\mathbf{s} \tag{19}$$

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$$V_i = \sum_{k=1}^{|\mathcal{N}_i|} \sum_{\mathcal{B} \in \mathcal{S}_k^{\mathcal{N}_i}} (-1)^k \int_{\mathbb{R}^2} \prod_{j \in \mathcal{B} \cup \mathcal{N}_i'} g_j \, d^2\mathbf{s} = \sum_{k=1}^{|\mathcal{N}_i|} \sum_{\mathcal{B} \in \mathcal{S}_k^{\mathcal{N}_i}} (-1)^k \gamma_{\mathcal{B} \cup \mathcal{N}_i'} \tag{23}$$

Closed-form marginal likelihood

The marginal semi-complete-data likelihood with half-normal detection function,

$$\mathcal{L}^{sc}(\boldsymbol{\theta}; \boldsymbol{\Omega}, \mathbf{X}, n) = \frac{D^n \exp(-D \cdot F_{\text{esa}})}{n!} \cdot F_{\text{det}_n} \tag{24}$$

$$= \frac{D^n}{n!} \exp\left(D \cdot \sum_{k=1}^{m} \sum_{\mathcal{B} \in \mathcal{S}_k} (-1)^k \gamma_{\mathcal{B}}\right) \cdot \prod_{i=1}^{n} \left(\gamma_{\mathcal{N}_i'} + \sum_{k=1}^{|\mathcal{N}_i|} \sum_{\mathcal{B} \in \mathcal{S}_k^{\mathcal{N}_i}} (-1)^k \gamma_{\mathcal{B} \cup \mathcal{N}_i'}\right) \tag{25}$$

where

$$\gamma_{\mathcal{B}} = g_0^{|\mathcal{B}|} \exp\left[\frac{|\mathcal{B}|}{2\sigma^2}\left(\|\mathbf{c}_{\mathcal{B}}\|^2 - \mu_{\mathcal{B}}\right)\right] \frac{2\pi\sigma^2}{|\mathcal{B}|} \tag{26}$$

and the term $\mathbf{c}_{\mathcal{B}}$ denotes the centroid of detectors defined by the set $\mathcal{B}$ and $\mu_{\mathcal{B}}$ denotes the mean squared Euclidean norm of the detector coordinates

Thank you!