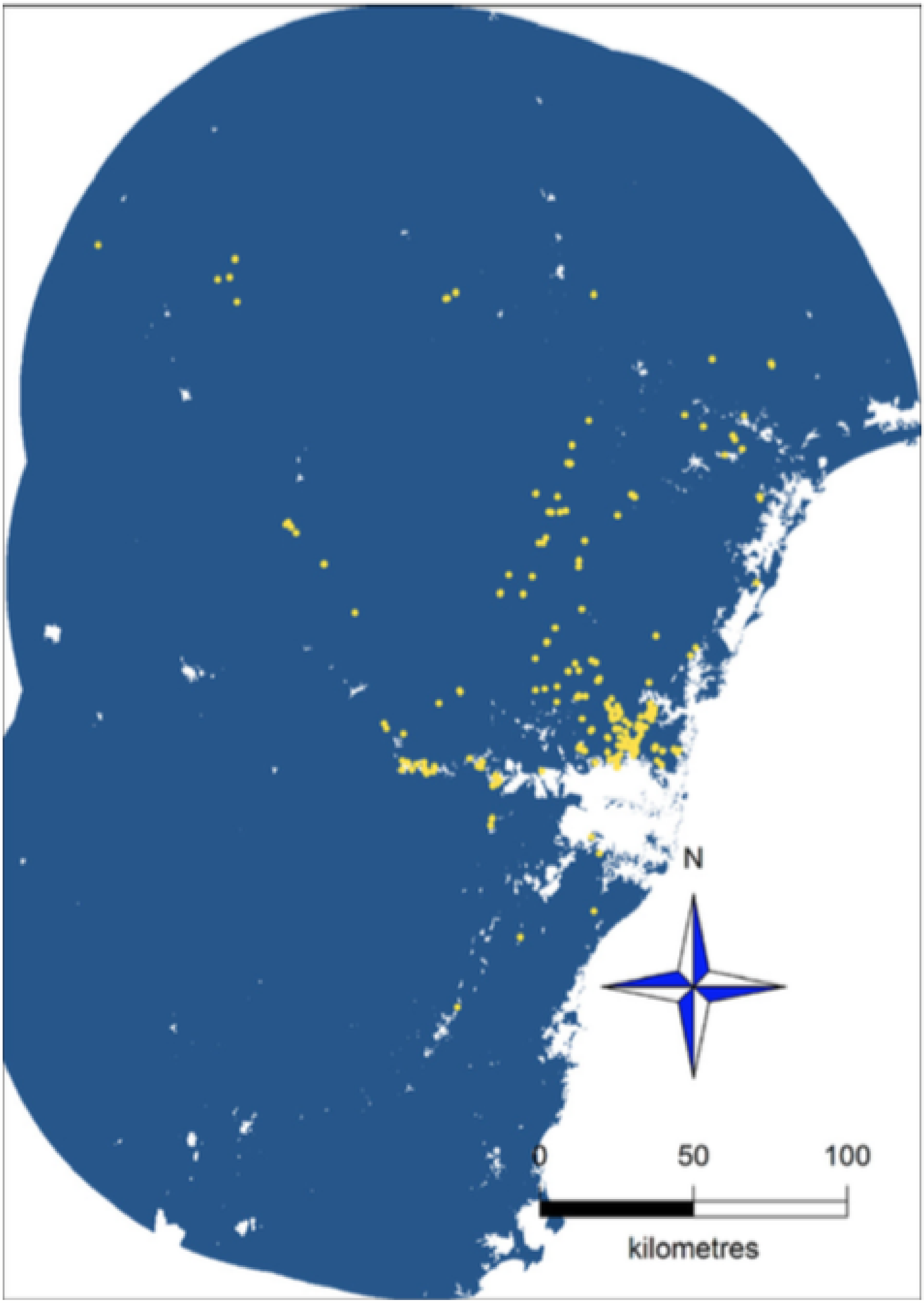
Spatial confounding in Cox process models

author: Wesley Brooks date: University of New South Wales

An example: ***Eucalyptus sparsifolia***

- Analysis method for presence-only data, e.g.: Locations of 230 presence-only *Eucalyptus sparsifolia* observa-tions within 100 km of the Greater Blue Mountains World Heritage Area (source: Renner *et al.*, 2015).


N
0
50
100
kilometres

Eucalyptus sparsifolia data

- Goal in analyzing the *Eucalyptus sparsifolia* data is a regression model that explains the effect of environmental covariates and maps the underlying distribution of *Eucalyptus sparsifolia*.
- Covariates:
- Rainfall
- Max, min annual average temperature
- Frequency of fires
- Soil type
- Distance from roads and urban areas

Eucalyptus sparsifolia models

- Source: Renner *et al.* (2015)
- Downweighted Poisson regression (DWPR)
- Log-Gaussian Cox process (LGCP)

Covariate	Est. coef (DWPR)	Est. coef (LGCP)
Intercept	-715.8	-11.6
Fire count	-3.4	-7.8
Min temp.	-29.2	-1.8
Max temp.	43.5	-0.1
Rainfall	326.4	-7.6

- It was proposed in the paper that the difference was down to **spatial confounding**.

Overview

- When analyzing presence-only data that exhibits spatial structure beyond that explained by covariates, a Cox process model may be useful.
- A Cox process may introduce a spatial random effect that is correlated with the covariates, leading to spatial confounding.
- We propose a method to implement a Cox process model that avoids spatial confounding by restricting the random effect to be orthogonal to the fixed effects.
- Also present a method to quickly estimate parameters of a Cox process via fixed-rank spatial effect and a variational approximation.

Outline

- Point process models
- Homogeneous Poisson
- Inhomogeneous Poisson
- Cox process
- Estimation
- Spatial confounding
- Spatial random effects
- Variational approximations and the `cox` package

- Simulation study

Poisson point process

- Homogeneous: assume a constant intensity λ on domain $\mathcal{D}$ with area $|\mathcal{D}|$.
- Number of points N is distributed as $N \sim \text{Pois}(\lambda|\mathcal{D}|)$.
- Location of points follows uniform distributon on the domain
- Inhomogeneous: intensity varies with location $\boldsymbol{s}$: $\lambda(\boldsymbol{s})$
- Now $N \sim \text{Pois}\{\int_{\mathcal{D}} \lambda(\boldsymbol{s})d\boldsymbol{s}\}$
- Point density is proportional to $\lambda(\boldsymbol{s})$.
- Likelihood of an observed data set from an inhomogeneous Poisson process:

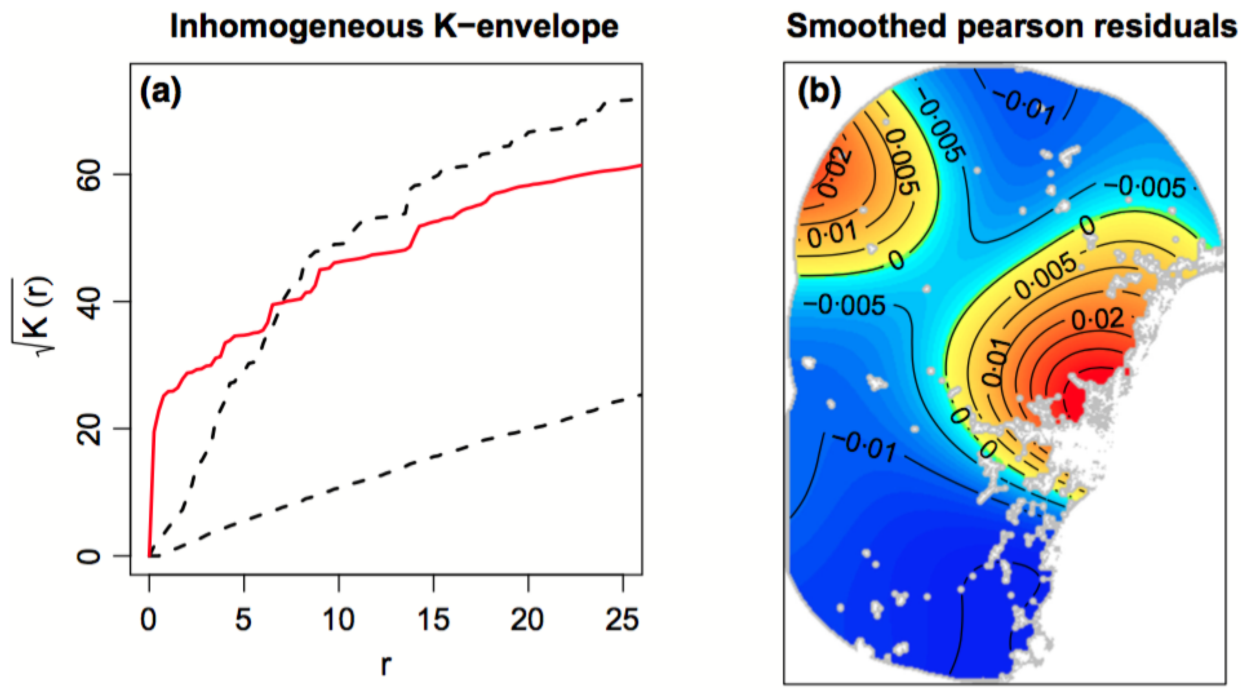
$$\mathcal{L}\{\lambda(\cdot)\} = \left\{\int_{\mathcal{D}} \lambda(\boldsymbol{s})d\boldsymbol{s}\right\}^N \exp\left\{-\int_{\mathcal{D}} \lambda(\boldsymbol{s})d\boldsymbol{s}\right\}/N!$$

Inhomogeneous Poisson process

- Regression: suppose the intensity is a function of some covariates $\boldsymbol{X}$.
- E.g., $\lambda(\boldsymbol{s}) = \exp\{\boldsymbol{x}'(\boldsymbol{s})\boldsymbol{\beta}\}$
- Renner *et al.* (2015) proposed a "down-weighted Poisson regression" (DWPR) for estimation.
- Uses numerical quadrature to evaluate the integral $\int_{\mathcal{D}} \lambda(\boldsymbol{s})d\boldsymbol{s}$.
- Numerical quadrature: discretize the domain into cells and sum the values in the cells (Berman and Turner, 1992).
- May require quadrature points numbering $10^4 - 10^6$.

Eucalyptus sparsifolia revisited

After modeling the disribution of *Eucalyptus sparsifolia* as an inhomogenous Poisson process, check for clustering: - Both K-function and the Pearson residuals indicate clustering not explained by the model (figures source: Renner *et al.*, 2015).



Cox process

- Often in regression models with spatially indexed observations, we include a factor to account for local similarity not explained by the covariates.
- Geostatistical correlation functions
- Conditional autoregressive models
- Cox process is an inhomogeneous Poisson process where
- $\lambda(\boldsymbol{s}) = \exp\{\boldsymbol{x}'(\boldsymbol{s})\boldsymbol{\beta} + \zeta(\boldsymbol{s})\}$
- $\boldsymbol{\zeta} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$
- $\Sigma_{ij} = C(\boldsymbol{s}_i, \boldsymbol{s}_j)$ is a spatial covariance function

Outline

- Point process models
- Homogeneous Poisson
- Inhomogeneous Poisson
- Cox process
- Estimation
- Spatial confounding
- Spatial random effects
- Variational approximations and the `cox` package
- Simulation study

Spatial confounding

- "*Adding spatially-correlated errors can mess up the fixed effect you love*" (Hodges and Reich, 2010)

- In general, $\boldsymbol{\zeta}$ may be correlated with columns of $\boldsymbol{X}$, which will affect the estimates $\hat{\beta}$
- Consider a linear mixed model, representing a conditional autoregressive (CAR) model with neighborhood matrix $\boldsymbol{Q}$:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{I}_n\boldsymbol{S} + \boldsymbol{\varepsilon}$$

- Where $\boldsymbol{S} \sim MVN\left(\boldsymbol{0}, (\tau_s\boldsymbol{Q})^{-1}\right)$
- Eigendecomposition of neighborhood matrix: $\boldsymbol{Q} = \boldsymbol{Z}\boldsymbol{D}\boldsymbol{Z}'$
- Now let $\boldsymbol{b} = \boldsymbol{Z}'\boldsymbol{S} \sim MVN\left(\boldsymbol{0}, (\tau_s\boldsymbol{D})^{-1}\right)$

Spatial confounding (source: Hodges and Reich, 2010)

- Rewrite the model:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{b} + \boldsymbol{\varepsilon}$$

- The eigenvalues $d_j, j = 1, \ldots, r$ are shrinkage parameters for components of the random effect.
- Any column of $\boldsymbol{Z}$ that is colinear with $\boldsymbol{X}$ and has little shrinkage is confounded with the fixed effects.
- Equivalently: any eigenvector of the spatial precision matrix that is colinear with $\boldsymbol{X}$ and has a small eigenvalue is confounded with the fixed effects.
- Confounding biases coefficient estimates.

Spatial confounding

- Spatial confounding has been identified in areal and geostatistical regression models, with various interpretations
- Unaccounted-for explanatory variable, in a geostatistical context (Paciorek, 2011)
- Covariates having spatial structure on the observational units (Hodges and Reich, 2010)
- Typical prescription has been to project the model's spatial random component into a subspace orthogonal to the covariates (Hodges and Reich, 2010; Hughes and Haran, 2013).
- What about point processes?

Spatial confounding in Cox process models

- Cox process models use $10^4 - 10^6$ quadrature points
- Covariance matrix of spatial random process may be $10^6 \times 10^6$
- Typical methods of accounting for spatial confounding are impractical for Cox process model
- Impractical to calculate the matrix of distances between locations for a geostatistical covariance model
- After projection orthogonal to the covariates, the neighborhood matrix for a CAR model would not be sparse
- Eigendecomposition, etc. are expensive
- This makes it impossible to assess degree of confounding

Outline

- Point process models
- Homogeneous Poisson
- Inhomogeneous Poisson
- Cox process
- Estimation
- Spatial confounding

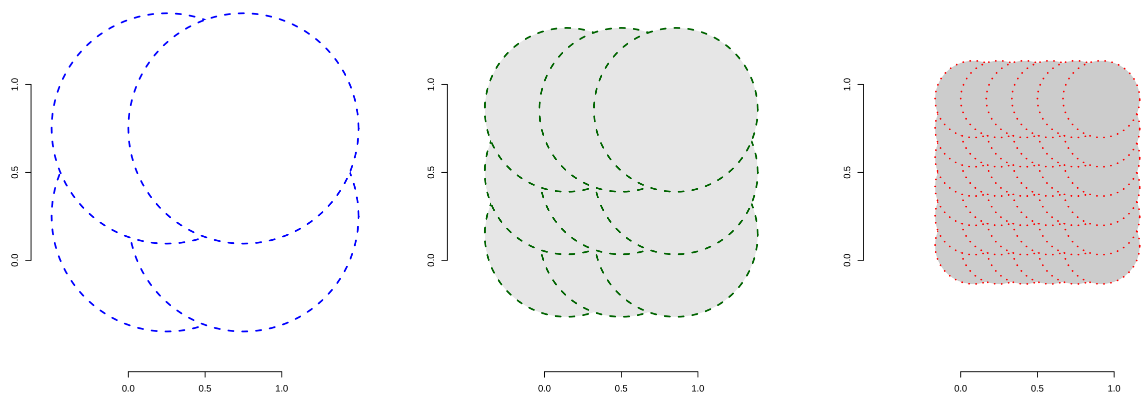
- Spatial random effects
- Variational approximations and the `cox` package
- Simulation study

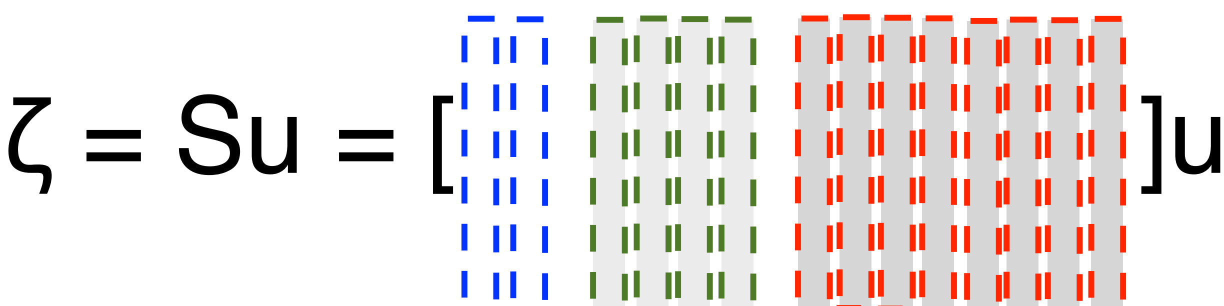
Spatial random effects

Cressie and Johannesson (2008) introduced "Fixed rank kriging"

- Approximates a Gaussian random field as the sum of a fixed number of basis functions
- Can use your favorite basis, multiresolution basis is recommended (Nychka *et al.*, 2002)
- E.g., multi-resolution bisquare functions
- Results in a low-rank random effect that is easily projected orthogonal to the fixed effects!

Multiresolution bisquares





$\boldsymbol{u} \sim MVN(\boldsymbol{0}, \tau \boldsymbol{I}_r)$

Outline

- Point process models

- Homogeneous Poisson
- Inhomogeneous Poisson
- Cox process
- Estimation
- Spatial confounding
- Spatial random effects
- Variational approximations and the `cox` package
- Simulation study

Marginal likelihood

- The likelihood to maximize is marginal to the random effects
- i.e., with the random effects "integrated out"
- For our model, this looks like

$$\pi(\boldsymbol{y}) = \int_U \pi(\boldsymbol{y}|\boldsymbol{u})\pi(\boldsymbol{u})\mathrm{d}\boldsymbol{u}$$

$$= \int_U \left[\int_{\mathcal{D}} \exp\left\{\boldsymbol{x}'(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) + \boldsymbol{S}\boldsymbol{u}\right\} d\boldsymbol{s}\right]^N \times$$

$$\exp\left[\int_{\mathcal{D}} \exp\left\{\boldsymbol{x}'(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) + \boldsymbol{S}\boldsymbol{u}\right\} d\boldsymbol{s}\right] / N! \times \left(\frac{\tau}{2\pi}\right)^{r/2} \exp\left(-\tau\frac{\boldsymbol{u}'\boldsymbol{u}}{2}\right)\mathrm{d}\boldsymbol{u}$$

Variational approximation

- Evaluating the integral analytically is not practical (possible?).
- Markov chain Monte Carlo is a common approach to approximating the integral, but slow.
- We propose a variational approximation to the marginal likelihood (Ormerod and Wand, 2010; Hui *et al.*, unpublished)
- The variational lower bound for an arbitrary density q is a result of Jensen's inequality:

$$\log \pi(\boldsymbol{y}) \geq \int_U q(\boldsymbol{u}) \log\left\{\pi(\boldsymbol{y}, \boldsymbol{u})/q(\boldsymbol{u})\right\} \mathrm{d}\boldsymbol{u}$$

Variational approximation

- Rewrite the likelihood lower bound:

$$\int_U \log\left\{\pi(\boldsymbol{y}, \boldsymbol{u})\right\} q(\boldsymbol{u})\mathrm{d}\boldsymbol{u} - \int_U \log\left\{q(\boldsymbol{u})\right\} q(\boldsymbol{u})\mathrm{d}\boldsymbol{u}$$

- Get the joint log-likelihood just by writing the hierarchical model, ignoring constants:

$$\log\{\pi(\boldsymbol{y}, \boldsymbol{u})\} = \sum_{i=1}^{n} \left[w_i \left\{y_i(\boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{S}_i'\boldsymbol{u}) - \exp(\boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{S}_i'\boldsymbol{u})\right\}\right] + r/2\log(\tau) - \boldsymbol{u}'\boldsymbol{u}/(2\tau)$$

- Assume that q is multivariate Gaussian with expectation $\boldsymbol{M}$ and variance $\boldsymbol{V}$.

Variational approximation

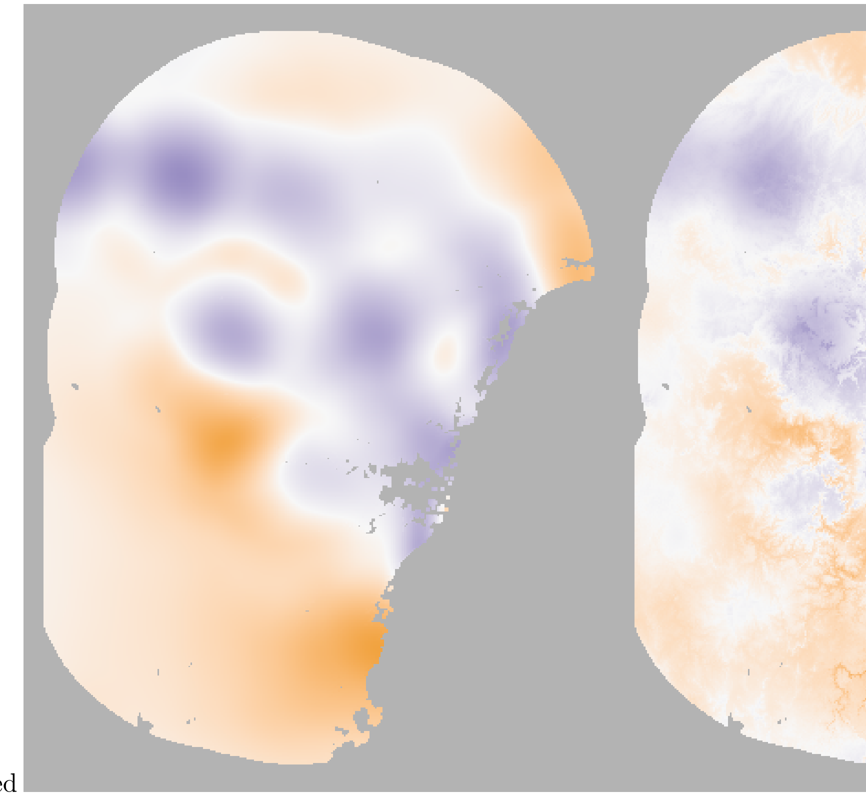
- Now finding the likelihood bound requires only that we calculate a few expectations with respect to a multivariate normal distribution:
- $\boldsymbol{u} \sim MVN(\boldsymbol{M}, \boldsymbol{V})$
- $E_u(\boldsymbol{u}'\boldsymbol{u}|\boldsymbol{M}, \boldsymbol{V})$
- $E_u(\boldsymbol{S}\boldsymbol{u}|\boldsymbol{M}, \boldsymbol{V})$
- $E_u\{\exp(\boldsymbol{S}\boldsymbol{u})|\boldsymbol{M}, \boldsymbol{V}\}$
- Maximize the lower bound by the method of conjugate gradient: requires only the first derivative of the lower bound
- `R` package is under active development, available via `devtools` from `github.com/wrbrooks/cox`.

Eucalyptus sparsifolia models

- Using a slightly different model than Renner *et al.* (2015) (no soil type), we estimate the coefficients:

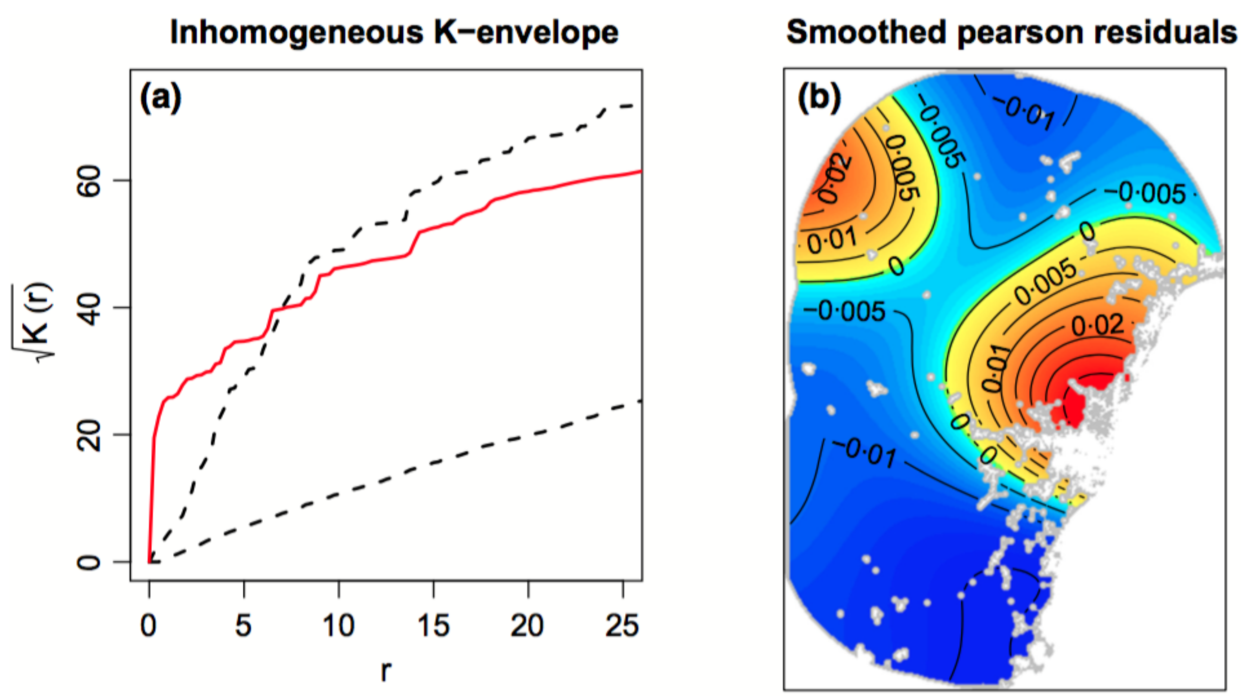
Covariate	DWPR	orthogonalized	nonorthogonalized
Intercept	-793.6	-222.8	95.8
Fire count	-6.6	-0.7	1.3
Min temp.	-25.9	-15.4	-6.7
Max temp.	47.8	14.4	-4.9
Rainfall	365.5	85.9	-40.6

Estimated random effects



- Left: not orthogonalized, right: orthogonalized

Recall:



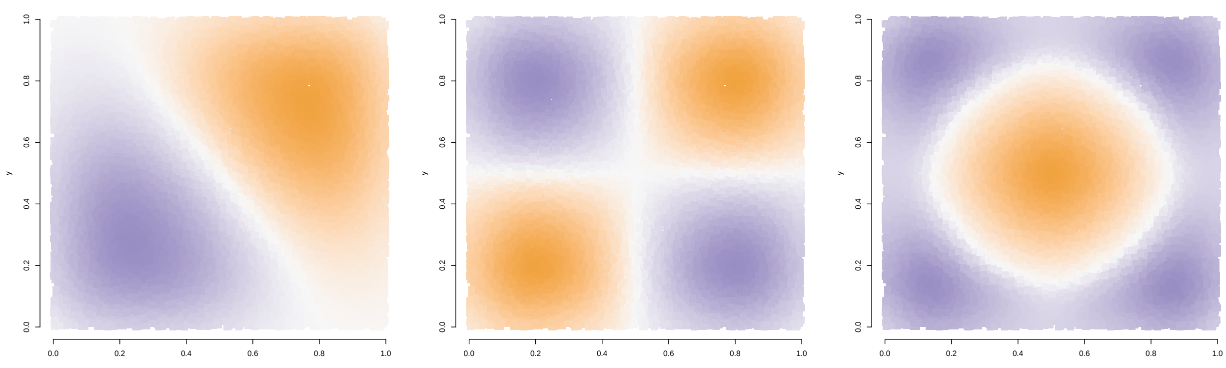
Outline

- Point process models
- Homogeneous Poisson
- Inhomogeneous Poisson
- Cox process
- Estimation
- Spatial confounding
- Spatial random effects
- Variational approximations and the `cox` package
- Simulation study

Simulaton study

- Design: generate covariates that are known to correlate with the random effect process
- Fix locations of the quadrature points and compute the matrix of multiresolution bisquares, $\boldsymbol{S}$
- Find the second, third, and fourth singular vectors of $\boldsymbol{S}$
- Model these singular vectors using a GAM, so that their values can be computed at any location
- These are the covariates.
- Simulate the linear predictor: $\eta = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\zeta}$
- $\boldsymbol{\beta} = (1, -0.2, 0)$
- $\boldsymbol{\zeta} \sim \text{GRF}(\sigma^2 = 0.5, \tau = 0.1)$ (exponential covariance)

Simulation study: covariates



Simulation study

- Scale the covariates
- `scale.X = 1, 2, 4` (Increase signal strength)
- Scale the random field
- `scale.re = 1, 0.5` (decrease noise)
- Alter the intercept
- β_0 `= 4, 5, 6` (Increase sample size)
- Estimate the regression parameters under multiresolution bisquares, both orthogonalized and nonorthogonalized.

Simlation results

- No apparent difference between orthogonalized, nonorthogonalized estimation!
- Bias
- MSE
- Estimated coefficient variance/confidence intervals
- Why?
- Relative to DWPR (no spatial effect) coefficient standard errors were smaller.

References

Berman, M. and Turner, T.R. (1992), "Approximating point process likelihoods with GLIM," *Journal of the Royal Statistical Society, Series C,* 41, 77-91.

Cressie, N., and Johannesson, G. (2008), "Fixed rank kriging for very large spatial data sets," *Journal of the Royal Statistical Society (Ser. B),* 70(1), 209-226.

Hodges, J.S. and Reich, B.J. (2010), "Adding spatially-correlated errors can mess up the fixed effect you love," *The American Statistician,* 64(4), 325-334.

Hughes, J. and Haran, M. (2013) "Dimension reduction and alleviation of confounding for spatial generalized linear mixed models," *Journal of the Royal Statistical Society, Series B,* 75(1), 139-159.

References

Hui, F.K.C., Warton, D.I., Ormerod, J.T., Haapaniemi, V., and Taskinen, S. (2015), "Variational approximations for generalized linear latent variable models," *unpublished.*

Ormerod, J.T., and Wand, M.P. (2010), "Explaining variational approximations," *The American Statistician,* 64(2), 140-153.

Renner, I.W., Elith, J., Baddeley, A., Fithian, W., Hastie, T., Phillips, S.J., Popovic, G., and Warton, D.I. (2015), "Point process models for presence-only analysis," *Methods in Ecology and Evolution,* 6, 366-379.