Estimating Abundance from Counts in Large Data Sets of Irregularly-Spaced Plots using Spatial Basis Functions

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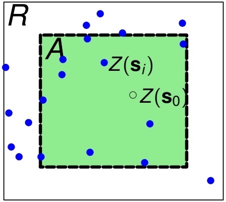
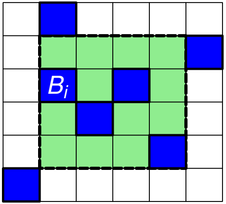
Introduction

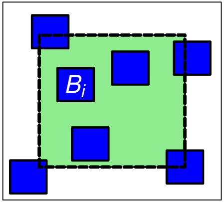
1) Block Kriging



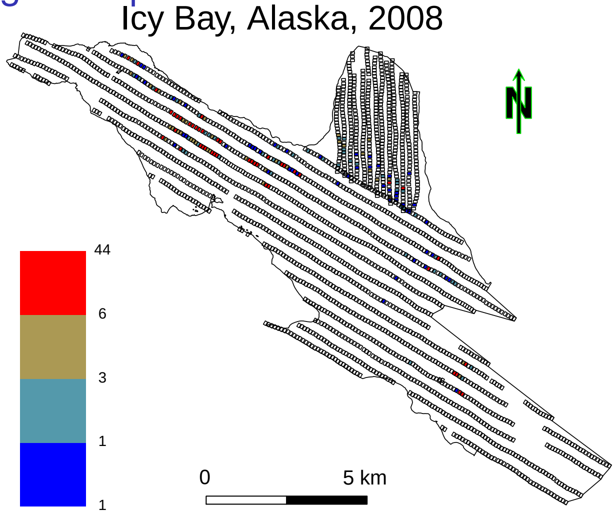
2)Block Prediction for Finite Populations on a Grid



3)Block Prediction for Finite Populations Irregularly Spaced



Motivating Example



Goals

An estimator that is:

- fast to compute, robust, and requires few modeling decisions, similar to classical survey methods,
- based only on counts within plots; actual spatial locations of animals are unknown,
- for the actual number of seals, not the mean of some assumed process that generated the data,
- have a variance estimator with a population correction factor that shrinks to zero as the proportion of the study area that gets sampled goes to one,
- unbiased with valid confidence intervals,
- able to accommodate nonstationary variance and excessive zeros throughout the area

Inhomogeneous Spatial Point Processes

$T(V)$ is the total number of points in planar region V

$$\lambda(\mathbf{s}) = \lim_{|dx|\to 0} \frac{E(T(dx))}{|dx|}$$

Expected abundance in $A \subseteq R$:

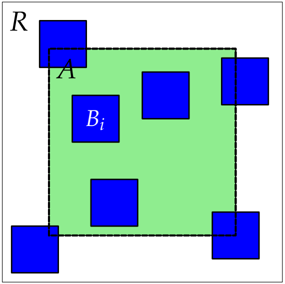
$$\mu(A) = \int_A \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$$

Abundance is assumed random

$$T(A) \sim \text{Poi}(\mu(A))$$

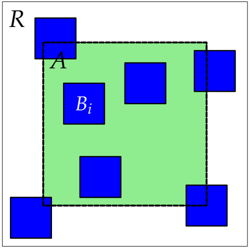
Resulting in an observed pattern $\mathcal{S}^+ = (\mathbf{s}_1, \ldots, \mathbf{s}_N)$

Outline of an Estimator



- $\mathcal{B} = \cup_{i=1}^{n}(B_i \cap A)$
- $\mathcal{U} \equiv \overline{\mathcal{B}} \cap A$
- $T(A) = T(\mathcal{B}) + T(\mathcal{U})$
- $T(\mathcal{U}) \sim \text{Poi}(\mu(\mathcal{U}))$
- $\mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$
- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $T(\mathcal{B}) \to T(A) \Rightarrow \widehat{T}(A) \to T(A)$

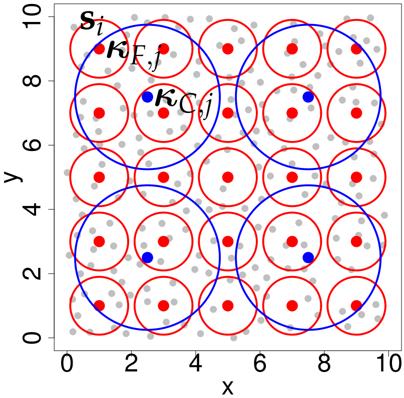
From IPP to Poisson Regression



- $Y(B_i) \sim \text{Poi}(\mu(B_i))$
- $\mu(B_i) = \int_{B_i} \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$
- Let $\mathbf{s}_i$ be centroid of B_i
- $\mu(B_i) \approx |B_i|\lambda(\mathbf{s}_i|\boldsymbol{\theta})$
- $\log(\mu(B_i)) = \log(|B_i|) + \log(\lambda(\mathbf{s}_i|\boldsymbol{\theta}))$
- $\log(\lambda(\mathbf{s}_i|\boldsymbol{\theta})) = \mathbf{x}(\mathbf{s}_i)'\boldsymbol{\beta}$

Now us spatial basis functions to generate $\mathbf{x}(\mathbf{s}_i)$

Spatial Basis Functions



- $C(h;\rho) = \exp(-h^2/\rho)$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{F,j})\|;\rho_F)$; $j = 2,\ldots,K_F+1$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{C,j})\|;\rho_C)$; $j = K_F+2,\ldots,K_F+K_C+1$

knot location: k-means clustering of dense grid of spatial coordinates

Fitting the Model

minimize minus the log-likelihood:

$$-\ell(\boldsymbol{\rho}, \boldsymbol{\beta}; \mathbf{y}) \propto \sum_{i=1}^{n} |B_i| \exp(\mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}) - y_i \log|B_i| - y_i \mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}$$

Two-part algorithm:

- Condition on ρ and use IWLS to estimate β (with offset for $|B_i|$, ala GLMs)
- optimize for ρ numerically

Back to the Estimator

- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $\widehat{T}(\mathcal{U}) = \mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) d\mathbf{u}$
- $\lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) = \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u})'\hat{\boldsymbol{\beta}})$

Approximate integral with dense grid of n_p points within $\mathbf{u}_j \in \mathcal{U}$.

$$\widehat{T}(A) = T(\mathcal{B}) + \sum_{j=1}^{n_p} |U_i| \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

where $|U_i|$ is a small area around each $\mathbf{u}_j$

Variance

$\text{MSPE}(\hat{T}(A)) = E[(\hat{T}(A) - T(A))^2; \boldsymbol{\beta}] = E[(\hat{T}(\mathcal{U}) - T(\mathcal{U}))^2; \boldsymbol{\beta}]$
Note: as $\mathcal{U} \cap A \to \varnothing \Rightarrow \text{MSPE}(\widehat{T}(A)) \to 0$
From IPP assumption: $\hat{T}(\mathcal{U})$ independent from $T(\mathcal{U})$.
Assuming unbiasedness, $E[(\hat{T}(\mathcal{U})] = E[T(\mathcal{U})]$,

$$\begin{aligned} \text{MSPE} &= \text{var}[T(\mathcal{U}); \boldsymbol{\beta}] + \text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}] \\ &= \mu(\mathcal{U}; \boldsymbol{\beta}) + \text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}] \end{aligned}$$

Now, what about $\text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}]$?

Variance

Recall delta method result: $\mathrm{var}(f(\mathbf{y})) \approx \mathbf{d}'\boldsymbol{\Sigma}\mathbf{d}$

Jay M. Ver Hoef (2012) Who Invented the Delta Method? The American Statistician, 66:2, 124-127

where $\mathrm{var}(\mathbf{y}) = \boldsymbol{\Sigma}$ and $d_i = \partial f(\mathbf{y})/\partial y_i$

$$d_i = \frac{\partial \hat{T}(\mathcal{U})}{\partial \beta_i} = \int_{\mathcal{U}} x_i(\mathbf{u}) \exp(\mathbf{x}(\mathbf{u})'\hat{\boldsymbol{\beta}}) d\mathbf{u} \approx \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

From Rathbun and Cressie, (1994), if $\hat{\beta}$ is MLE,

$$\hat{\boldsymbol{\Sigma}} = \left[\sum_{i=1}^{n} \int_{B_i} \mathbf{x}(\mathbf{s})\mathbf{x}(\mathbf{s})' \exp(\mathbf{x}(\mathbf{s})'\hat{\boldsymbol{\beta}}) d\mathbf{s} \right]^{-1} \approx \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}}) \right]^{-1}$$

if $|B_i| = |B| \; \forall \; i$.

Rathbun, S. L. and Cressie, N. (1994), “Asymptotic Properties of Estimators for the Parameters of Spatial Inhomogeneous Poisson Point Processes,” Advances in Applied Probability, 26, 122-154.

Summary

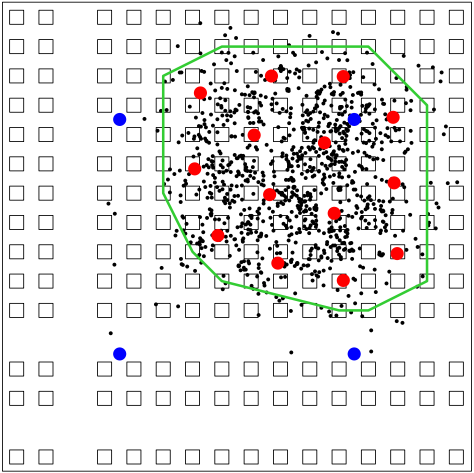
$$\widehat{T}(A) = T(\mathcal{B}) + \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

$$\begin{aligned}\widetilde{\mathrm{var}}(\widehat{T}(A)) = \quad & \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}}) + \\ & \mathbf{d}' \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}}) \right]^{-1} \mathbf{d}\end{aligned}$$

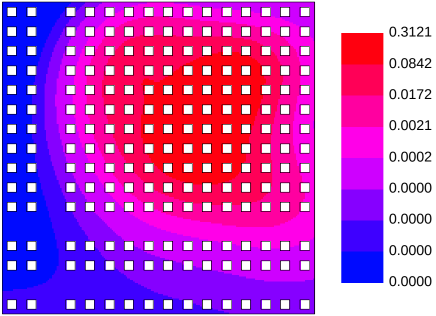
where

$$d_i = \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

Simulated Example

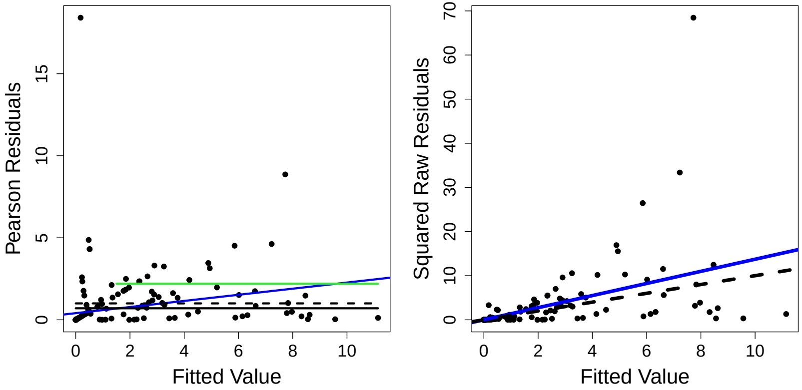


Simulated Example



True abundance was 1079
Estimated abundance was 1143 with standard error 62

Residuals Plots



Overdispersion Estimators

- The traditional estimator:

$$\omega_{OD} = \max\left(1, \frac{1}{n-r}\sum_{i=1}^{n}\frac{(y_i-\phi_i)^2}{\phi_i}\right)$$

  where r is the rank of $\mathbf{X}$.
- Weighted regression estimator:

$$\omega_{WR} = \max\left(1, \arg\min_{\omega}\sum_{i=1}^{n}\sqrt{\phi_i}[(y_i-\phi_i)^2-\omega\phi_i]^2\right),$$

  where $\sqrt{\phi_i}$ were the weights

Overdispersion Estimators

- Trimmed Mean:

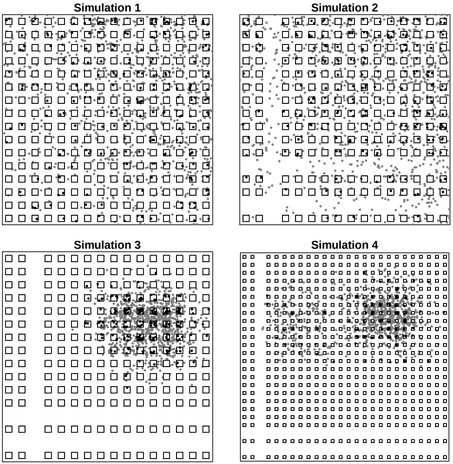
$$\omega_{TG}(p) = \max\left(1, \frac{1}{n - \lfloor np \rfloor - r} \sum_{i=\lfloor np \rfloor + 1}^{n} \frac{(y_{(i)} - \phi_{(i)})^2}{\phi_{(i)}}\right)$$

where $0 \le p \le 1$, $y_{(i)}$ and $\phi_{(i)}$ are ordered values, and $\lfloor x \rfloor$ rounds x down to the nearest integer.

Adjusted Variance Estimators

- $\widehat{\text{var}}_{OD}(\widehat{T}(A)) = \omega_{OD}\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{WR}(\widehat{T}(A)) = \omega_{WR}\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{TG}(\widehat{T}(A); p) = \omega_{TG}(p)\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{TL}(\widehat{T}(A); p) = \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}}) \times$ $\max(1, \omega_{TG}(p) I(\exp(\mathbf{x}(\mathbf{s}_j)'\hat{\boldsymbol{\beta}}) \geq \phi_{(\lfloor np \rfloor)}) +$ $((1-p)\omega_{TG}(p) + p) \times$ $\mathbf{d}' \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}}) \right]^{-1} \mathbf{d}$
  where $I(\cdot)$ is the indicator function

Simulations



Simulation Experiment 1

		Knots			
	SRS	$K_C = 3$ $K_F = 8$	$K_C = 4$ $K_F = 14$	$K_C = 5$ $K_F = 20$	$K_C = 6$ $K_F = 26$
bias	6.425	-1.277	-9.735	7.048	5.941
RMSPE	58.060	57.493	59.036	58.243	58.038
CI90	0.914	0.898	0.883	0.892	0.895
CI90$_{OD}$		0.918	0.927	0.897	0.904
CI90$_{WR}$		0.900	0.892	0.895	0.895
CI90$_{TG}$		0.914	0.920	0.939	0.957
CI90$_{TL}$		0.906	0.906	0.917	0.930
fail rate	0.000	0.000	0.000	0.000	0.000

Simulation Experiment 2

	SRS	Knots			
		$K_C = 3$ $K_F = 8$	$K_C = 4$ $K_F = 14$	$K_C = 5$ $K_F = 20$	$K_C = 6$ $K_F = 26$
bias	79.234	-1.333	-7.632	14.511	13.856
RMSPE	104.979	66.347	68.846	68.311	68.527
CI90	0.726	0.876	0.860	0.891	0.889
CI90$_{OD}$		0.908	0.928	0.901	0.902
CI90$_{WR}$		0.888	0.868	0.894	0.891
CI90$_{TG}$		0.902	0.903	0.952	0.980
CI90$_{TL}$		0.895	0.880	0.927	0.940
fail rate	0.000	0.000	0.000	0.000	0.001

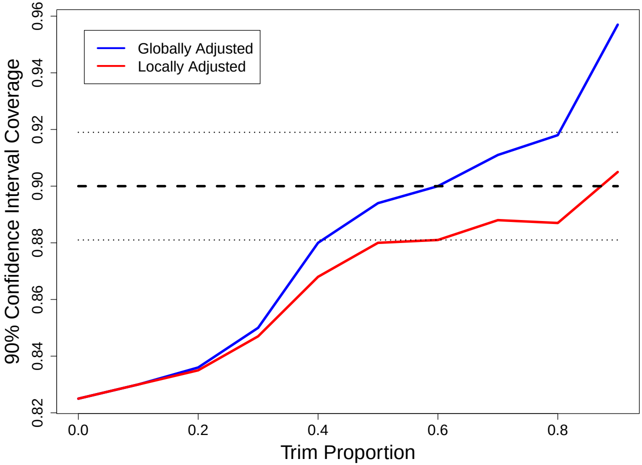
Simulation Experiment 3

		Knots			
	SRS	$K_C = 3$ $K_F = 8$	$K_C = 4$ $K_F = 14$	$K_C = 5$ $K_F = 20$	$K_C = 6$ $K_F = 26$
bias	214.816	-2.389	-4.365	-2.919	-1.637
RMSPE	235.713	79.207	79.250	79.285	80.175
CI90	0.774	0.775	0.772	0.781	0.777
CI90_{OD}		0.801	0.783	0.789	0.782
CI90_{WR}		0.918	0.906	0.865	0.837
CI90_{TG}		0.923	0.930	0.929	0.946
CI90_{TL}		0.871	0.883	0.878	0.903
fail rate	0.000	0.000	0.000	0.000	0.018

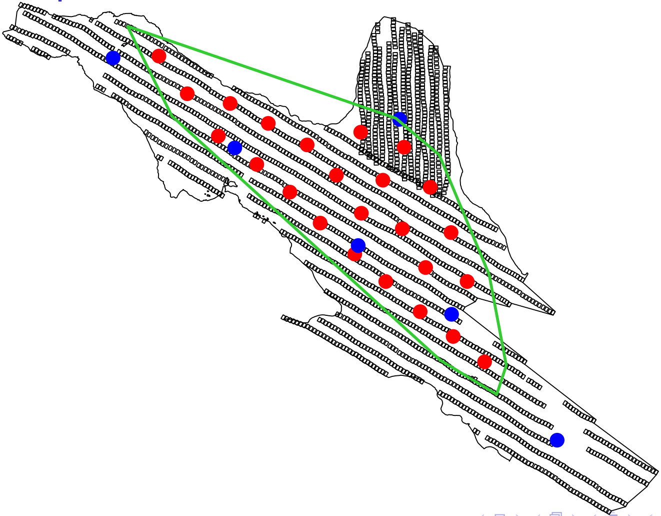
Simulation Experiment 4

		Knots			
	SRS	$K_C = 3$ $K_F = 8$	$K_C = 5$ $K_F = 16$	$K_C = 7$ $K_F = 24$	$K_C = 9$ $K_F = 32$
bias	148.523	5.179	3.440	7.287	14.629
RMSPE	163.516	60.403	61.021	62.102	64.136
CI90	0.834	0.831	0.825	0.826	0.833
CI90_{OD}		0.844	0.831	0.827	0.833
CI90_{WR}		0.939	0.928	0.923	0.913
CI90_{TG}		0.929	0.919	0.907	0.920
CI90_{TL}		0.893	0.892	0.884	0.886
fail rate	0.000	0.000	0.000	0.000	0.242

Effect of p in Trimmed Overdispersion Estimator



Real Example



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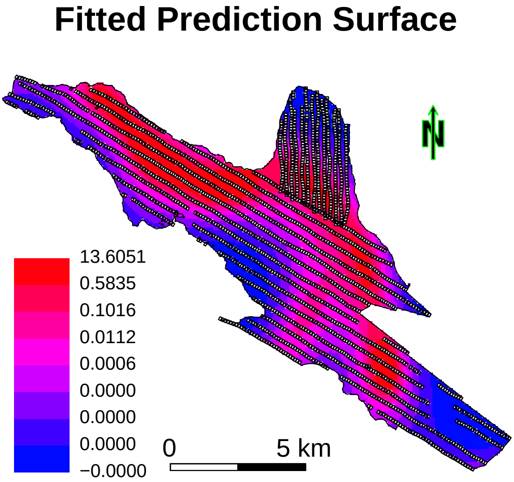

Real Example

```
sealDensity <- sum(plots@data[, "counts"])/(sCSout$propSurveyed * totalArea)
sealDensity * totalArea


## [1] 3960


summary(sCSout)


##
## Estimates:
##
## Total:
## [1] 4068
## Standard Errors:
##      SE SE.ODTrad SE.ODTrimGlobal SE.ODTrimLocal SE.ODRegr
## 1 113.7      1379             236          198.5     403.1
##
##
## Range Parameters:
##   coarseScale fineScale
## 1        9516      2974
##
##
## Proportion Surveyed:
## [1] 0.253
```

Recall the Goals

An estimator that is:

- fast to compute, robust, and requires few modeling decisions, similar to classical survey methods,
- based only on counts within plots; actual spatial locations of animals are unknown,
- for the actual number of seals, not the mean of some assumed process that generated the data,
- have a variance estimator with a population correction factor that shrinks to zero as the proportion of the study area that gets sampled goes to one,
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